Solution of incomplete cubic equation through development of its geometric model

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Abstract. We present a method of solution of an incomplete cubic equation of type $x^3 + px + q = 0$ with real values of the coefficients $p > 0$ and $q < 0$ through development of its geometric model in the space with dimensionality equal to the degree of this equation, i.e. in the three-dimensional space, and we expand this method to the whole range of incomplete cubic equations with real values of the coefficients.

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Assume that a cube with an edge of x is embedded in a cube with an edge of c so that each of the cubes can be obtained through scaling of other cube with respect to its center (fig.1, a). The region of space limited with the surfaces of the two cubes can be made of the six parallelepipeds with edges of x, c and b and the eight cubes with an edge of b (fig.1, b).

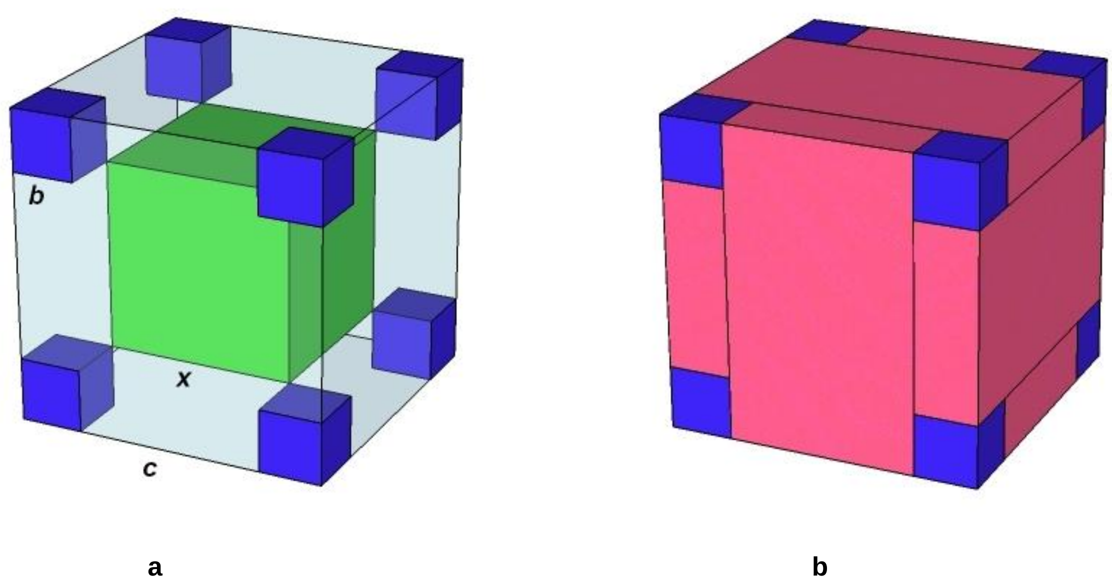

Fig.1. The geometric model of an incomplete cubic equation of type $x^3 + px + q = 0$ with real values of the coefficients $p > 0$ and $q < 0$ in the three-dimensional space. **(a)** The parallelepipeds with edges of x, c and b are not depicted. **(b)** The complete geometric model of an incomplete cubic equation.

In this case real values of x, c and b are limited with the condition

$$(x > 0) \wedge (c > 0) \wedge \left(0 < b < \frac{c}{2}\right) \qquad (1)$$

and are connected by the relations

$$c = x + 2b, \qquad (2)$$

$$c^3 = x^3 + 6bcx + 8b^3, \qquad (3)$$

where c^3 – volume of the bigger cube,
x^3 – volume of the embedded cube,

$6bcx$ – total volume of the six parallelepipeds with edges of x, c and b,
$8b^3$ – total volume of the eight cubes with an edge of b.

We rewrite the relations (2) and (3) in the following way

$$x = c - 2b, \tag{4}$$

$$x^3 + 6bcx + 8b^3 - c^3 = 0. \tag{5}$$

We make two substitutions in the relation (5). Let

$$6bc = p, \tag{6}$$

$$8b^3 - c^3 = q. \tag{7}$$

In this case the relation (5) is rearranged to

$$x^3 + px + q = 0. \tag{8}$$

From the condition (1) and the relations (6) and (7) it follows that for the incomplete cubic equation (8) it is possible to develop its geometric model in the space with dimensionality equal to the degree of this equation, i.e. in the three-dimensional space, with real values of its coefficients

$$(p > 0) \wedge (q < 0), \tag{9}$$

here the length of an edge of the embedded cube is equal to the positive real root of the incomplete cubic equation (8).

We rearrange the relations (6) and (7) to

$$(2b)^3(-c)^3 = -\left(\frac{p}{3}\right)^3,$$

$$(2b)^3 + (-c)^3 = q$$

and make two substitutions in them. Let

$$(2b)^3 = \alpha_1, \tag{10}$$

$$(-c)^3 = \alpha_2. \tag{11}$$

Then we have

$$\alpha_1\alpha_2 = -\left(\frac{p}{3}\right)^3,$$

$$\alpha_1 + \alpha_2 = q.$$

The variables α_1 and α_2 are roots of the quadratic equation

$$\alpha^2 - q\alpha - \left(\frac{p}{3}\right)^3 = 0$$

and respectively equal to

$$\alpha_1 = \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3},$$

$$\alpha_2 = \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

In accordance with the relations (10) and (11) we have

$$(-c)^3 = \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, \qquad c = -\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}},$$

$$(2b)^3 = \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, \qquad 2b = \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

In accordance with the relation (4) the length of an edge of the embedded cube and the positive real root of the incomplete cubic equation (8) accounting the condition (9) are equal to

$$x = -\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}. \qquad (12)$$

Let's cancel the limitation (1) acting on values of x, c and b, and assume that $x \in \mathbb{R},\ c, b \in \mathbb{C}$. In this case the relations (2) and (3) preserve truth with $x \in \mathbb{R},\ c, b \in \mathbb{C}$. It follows that the relation (3) is equivalent to the relation (8) with $x \in \mathbb{R},\ c, b \in \mathbb{C},\ p, q \in \mathbb{R}$. That is why the relation (12) in accordance with the theory of incomplete cubic equation is the formula of one real root of the incomplete cubic equation (8) when it has not more than two real roots and the formula of three real roots of this equation when it has these roots.

The second real root of the incomplete cubic equation (8) when it has two real roots is found from the relation

$$x_2 = \frac{x_1}{2},$$

where x_1 is defined by the relation (12), and equal to

$$x_2 = \sqrt[3]{\frac{q}{2}}.$$

Two complex roots of the incomplete cubic equation (8) when it has these roots are found by the known procedure using the relation (12) and equal to

$$x_{2,3} = \frac{1}{2}\left[\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\right] \pm \frac{\sqrt{3}i}{2}\left[\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\right].$$

It should be mentioned, that the presented method of solution of the incomplete cubic equation is close to the Cardano's method discussed in [1], however unlike the latter the discussed method naturally comes from pictorial geometric presentation of this equation.

References

[1] Cardano, Gerolamo (1545). Ars magna or The Rules of Algebra, Dover Publications (1993).